

# On the viscosity to entropy density ratio for unitary Bose and Fermi Gases

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## Abstract

We calculate the ratio of the viscosity to the entropy density for both Bose and Fermi gases in the unitary limit using a new approach to the quantum statistical mechanics of gases based on the S-matrix. In the unitary limit the scattering length diverges and the S-matrix equals  $-1$ . For the fermion case we obtain  $\eta/s > 4.7$  times the proposed lower bound of  $\hbar/4\pi k_B$  which came from the AdS/CFT correspondence for gauge theories, consistent with the most recent experiments. For the bosonic case we present evidence that the gas undergoes a phase transition to a strongly interacting Bose-Einstein condensate, and is a more perfect fluid, with  $\eta/s > 1.3$  times the bound.

## I. INTRODUCTION

One measure of the perfection of a fluid is its viscosity. In recent years new insights on such properties came from string theory, more precisely the AdS/CFT correspondence[1, 2]. This correspondence relates a conformally invariant (scale invariant) strongly coupled gauge theory in  $d + 1$  space-time dimensions to a gravitational dual in one higher spatial dimension. By studying black hole solutions in the higher dimensional theory, one can study finite temperature and density properties of the lower dimensional quantum field theory. In its original version, the conformal quantum field theory is a certain  $N = 4$  supersymmetric gauge theory, which is conformally invariant for all couplings since the beta function vanishes. Although such a supersymmetric gauge theory does not describe nature as we currently understand it, the AdS/CFT correspondence is nevertheless very useful for thinking about these difficult problems in new ways and inspiring the study of specific properties that were hardly considered before. A prominent example is the ratio of the shear viscosity  $\eta$  to the entropy density  $s$ . In natural units with  $\hbar = k_B = 1$ , it is a dimensionless quantity. Using the AdS/CFT correspondence it was found that the supersymmetric gauge theory had  $\eta/s = 1/4\pi$ . It was conjectured that this value represents a lower bound[2], i.e.

$$\frac{\eta}{s} \geq \frac{1}{4\pi} \frac{\hbar}{k_B} \quad (1)$$

The observed  $\eta/s$  for the quark-gluon plasma, studied via heavy ion collisions, is about 5 times this bound[3]. Theoretical estimates for the quark-gluon plasma tend to be lower, but consistent with the bound.

Note that the bound (1) does not depend on the speed of light. Thus, the natural question arises: “Do non-relativistic condensed matter systems respect the conjectured bound, and which class of fluids has the smallest  $\eta/s$ ?” For the relativistic fluids studied with the AdS/CFT correspondence, the scale invariance plays an im-

portant role. It has thus been proposed that the unitary Fermi gas may be the most perfect fluid[4–7], in part because it is scale invariant. In the so-called unitary limit of a quantum Bose or Fermi gas, the scattering length  $a$  diverges. This occurs at a fixed point of the renormalization group, thus these systems provide interesting examples of interacting, scale-invariant theories with dynamical exponent  $z = 2$ , i.e. non-relativistic. They can be realized experimentally by tuning the scattering length to  $\pm\infty$  using a Feshbach resonance. (See for instance [8, 9] and references therein.) They are also thought to occur at the surface of neutron stars. These systems have attracted much theoretical interest[10–24].

Because of the scale-invariance, the only length scales in the problem are based on the density  $n^{1/3}$ , and the thermal wavelength  $\lambda_T = \sqrt{2\pi/mT}$ . Equivalently, the only energy scales are the chemical potential  $\mu$  and the temperature  $T$ . The problem is challenging since there is no small parameter to expand in such as  $na^3$ . Any possible critical point must occur at a specific value of  $x = \mu/T$ . This can be translated into universal values for  $n_c\lambda_T^3$ , or for fermions universal values for  $T_c/T_F$  where  $\epsilon_F = k_B T_F$  is the Fermi energy. For instance the critical point of an ideal Bose gas is the simplest example, where  $n_c\lambda_T^3 = \zeta(3/2) = 2.61$ .

The models considered are the simplest models of non-relativistic bosons or fermions with quartic interactions. The bosonic model is defined by the action for a complex scalar field  $\phi$ .

$$S = \int d^3\mathbf{x} dt \left( i\phi^\dagger \partial_t \phi - \frac{|\vec{\nabla} \phi|^2}{2m} - \frac{g}{4} (\phi^\dagger \phi)^2 \right) \quad (2)$$

For fermions, due to the fermionic statistics, one needs at least a 2-component field  $\psi_{\uparrow,\downarrow}$ :

$$S = \int d^3\mathbf{x} dt \left( \sum_{\alpha=\uparrow,\downarrow} i\psi_\alpha^\dagger \partial_t \psi_\alpha - \frac{|\vec{\nabla} \psi_\alpha|^2}{2m} - \frac{g}{2} \psi_\uparrow^\dagger \psi_\uparrow \psi_\downarrow^\dagger \psi_\downarrow \right) \quad (3)$$

In both cases, positive  $g$  corresponds to repulsive interactions. The bosonic theory

only has a  $U(1)$  symmetry. The fermionic theory on the other hand has the much larger  $SO(5)$  symmetry. This is evident from the work[24] which considered an  $N$ -component version with  $Sp(2N)$  symmetry, and noting that  $Sp(4) = SO(5)$ .

The original AdS/CFT conjecture was for a specific, relativistic, and supersymmetric gauge theory. There have been some proposals to use the AdS/CFT correspondence to learn about non-relativistic systems[25–28]. One difficulty is that the conformal symmetry of relativistic systems is larger than the Schrödinger symmetry of non-relativistic systems, so the black hole solutions on the gravity side are less obvious. Also, given a black hole geometry, it remains unclear which condensed matter system it is dual to. Thus far, the AdS/CFT approaches for non-relativistic systems lead to  $\eta/s = 1/4\pi$ . Recent experimental work on the unitary Fermi gas reports values of  $\eta/s$  about 4 – 5 times the bound[32], thus it seems unlikely that a gravity dual exists that corresponds exactly to the unitary Fermi gas. Nevertheless, it is hoped that one can still discover some general, model-independent properties, in the same way that AdS/CFT for supersymmetric Yang-Mills provided insights into QCD.

For the remainder of this paper we will describe a novel, but more conventional approach to the problem[29]. The main approximation made is that we only consider 2-body interactions, and consistently resum their contributions to the free energy via an integral equation. The 2-body interactions are expressed in terms of the zero-temperature S-matrix, which can be calculated exactly. The viscosity is calculated in a kinetic, semi-classical approach. The calculation predicts a minimum  $\eta/s$  of 4.7 times the conjectured lower bound. This is consistent with the most recent experiments[32], which suggests our approximation is a good one.

Theoretical studies have mainly focussed on the fermionic case, and for the most part at zero temperature, which is appropriate for a large Fermi energy. The bosonic case has been less studied, since a homogeneous bosonic gas with attractive interac-

tions is thought to be unstable against mechanical collapse, and the collapse occurs before any kind of BEC. The situation is actually different for harmonically trapped gases, where BEC can occur[33]. However studies of the homogeneous bosonic case were based on a small, negative scattering length[34–37], and it is not clear that the conclusions reached there can be extrapolated to the unitary limit. Since the density of collapse is proportional to  $1/a$ [35], extrapolation to infinite scattering length suggests that the gas collapses at zero density, which seems unphysical, since the gas could in principle be stabilized at finite temperature by thermal pressure. One can also point out that in the van der Waals gas, the collapse is stabilized by a finite size of the atoms, which renders the compressibility finite. In the unitary limit, there is nothing to play such a role. In the sequel we will present evidence that the unitary Bose gas undergoes BEC when  $n\lambda_T^3 \approx 1.3$ . This lower value compared to the free case is consistent with the attractive interactions. For this bosonic case, the minimum  $\eta/s$  predicted in [29] was only 1.3 times the bound, and thus may be a better candidate than unitary fermions for the most perfect strongly interacting fluid.

## II. S-MATRIX, RENORMALIZATION GROUP AND SCATTERING LENGTH

In this section we describe the renormalization group fixed point, the bound state, and the scattering length. The interplay between all of these properties is most clearly seen from the S-matrix. These results are well-known and can be found in the book [30] for instance. We include them here for completeness and in order to consistently describe all of our conventions.

The free versions of the models described in the introduction have a scale invariance with dynamical exponent  $z = 2$ , i.e. are invariant under

$$t \rightarrow \Lambda^{-2}t, \mathbf{x} \rightarrow \Lambda^{-1}\mathbf{x}$$

As we now explain, the models possess a renormalization group fixed point, i.e. quantum critical point, where they have the same scale invariance. The renormalization group behavior can be inferred from the coupling constant dependence of the S-matrix. Consider first the single boson; the differences for two-component fermions will be described at the end of this section. The S-matrix can be calculated exactly by summing multi-loop ladder diagrams. By the Galilean invariance, the two-body S-matrix depends only on the difference of the incoming momentum of the two particles  $\mathbf{k}, \mathbf{k}'$ :

$$S(|\mathbf{k} - \mathbf{k}'|) = \frac{16\pi/mg_R - i|\mathbf{k} - \mathbf{k}'|}{16\pi/mg_R + i|\mathbf{k} - \mathbf{k}'|} \quad (4)$$

Unitarity of the S-matrix amounts to  $S^*S = 1$ .

The momentum space integrals for the higher loop corrections are divergent and an upper cut-off  $\Lambda$  must be introduced. In the above expression,  $g_R$  is the renormalized coupling:

$$\frac{1}{g_R} = \frac{1}{g} + \frac{m\Lambda}{4\pi^2} \quad (5)$$

Defining  $g = \hat{g}/\Lambda$ , where  $\hat{g}$  is dimensionless, and requiring  $g_R$  to be independent of  $\Lambda$  gives the beta-function:

$$\frac{d\hat{g}}{d\ell} = -\hat{g} - \frac{m}{4\pi^2}\hat{g}^2 \quad (6)$$

where  $\ell = -\log \Lambda$  is the logarithm of a length scale. The above beta function is exact since it was calculated from the exact S-matrix. One thus sees that the theory possesses a fixed point at the negative coupling  $g_* = -4\pi^2/m\Lambda$ .

We turn now to the scattering length  $a$ . From the above expression for the S-matrix one can infer the scattering amplitude and compute the total cross-section  $\sigma$ . Equating  $\sigma = \pi a^2$  gives

$$a(k) = \frac{m}{2\pi} \frac{g_R}{\sqrt{1 + (mg_R k/8\pi)^2}} \quad (7)$$

where here  $k$  is the momentum of one of the particles in the center of mass frame. If  $a(k)$  is measured at very small momentum transfer  $|\mathbf{k} - \mathbf{k}'| \approx 0$ , this leads to the definition of the scattering length

$$a = \frac{mg_R}{2\pi} = \frac{mg}{2\pi(1 - g/g_*)} \quad (8)$$

One sees that scattering length diverges at precisely the fixed point  $g = g_*$ . Note the S-matrix becomes  $S = -1$ . The scattering length  $a \rightarrow \pm\infty$ , depending on from which side  $g_*$  is approached. When  $g = g_*^-$ , i.e. just less than  $g_*$ , then  $a \rightarrow \infty$ , whereas when  $g = g_*^+$ ,  $a \rightarrow -\infty$ .

Finally, we turn to the bound state. The S-matrix (4) has a pole at  $k = 16\pi i/mg_R$ . Since physical bound states correspond to poles at  $\text{Im}(k) > 0$ , the bound state exists only for  $g$  below  $g_*$ . The energy of this bound state is

$$E_{\text{bound-state}} = -\frac{128\pi^2}{m^3 g_R^2} \quad (9)$$

Note that at the fixed point where  $g_R$  diverges, the energy of the bound state goes to zero as it should, since it disappears beyond this point.

Consider now two-component fermions with the action in the Introduction. The relative normalizations of the coupling  $g$  were chosen such that the beta function is the same for both the boson versus fermion cases. The S-matrix eq. (4) here represents the scattering of two fermions of opposite spin. Thus, the fermion case also has a diverging scattering length at the fixed point. In the fermionic context, the coupling  $g_*$  is the boundary of the so-called BEC/BCS crossover. On the BCS side just above  $g_*$ , the scattering length is negative, which implies effectively attractive interactions. Here it is believed that at low enough temperatures there is a phase transition to a strongly interacting version of superconductivity. For  $g < g_*$ , the scattering length is positive, signifying repulsive interactions, and a bound state exists. This bosonic bound state can undergo Bose-Einstein condensation, hence

this region is referred to as the BEC side. The physics is expected to be smooth as one crosses  $g_*$ . In the treatment of the thermodynamics below, we will work on the BCS side since here there is no need to incorporate a bound state into the thermodynamics.

### III. THERMODYNAMICS AT THE QUANTUM CRITICAL POINT

At the quantum critical point, the only energy scales in the problem are the chemical potential  $\mu$  and the temperature  $T = 1/\beta$ . This implies some universal scaling forms for the various thermodynamic functions[13]. The free energy density has the form

$$\mathcal{F} = -\zeta(5/2)T \lambda_T^{-3} c(\mu/T) \quad (10)$$

where  $\zeta$  is Riemann's zeta function and  $\lambda_T = \sqrt{2\pi/mT}$  is the thermal wavelength. The scaling function  $c$  is only a function of  $x \equiv \mu/T$ . With the above normalization, a single free boson has  $c = 1$  in the limit of  $x \rightarrow 0$ . It is also convenient to define the scaling function  $q$ , which is a measure of the quantum degeneracy, in terms of the density  $n$  as follows:

$$n\lambda_T^3 = q \quad (11)$$

The two scaling functions  $c$  and  $q$  are of course related since  $n = -\partial\mathcal{F}/\partial\mu$ , which leads to  $q = \zeta(5/2)c'$ , where  $c'$  is the derivative of  $c$  with respect to  $x$ . Henceforth  $b'$  will always denote the derivative of a function  $b(x)$  with respect to  $x$ .

The approach to the statistical mechanics of particles developed in complete generality in [38] is based on the S-matrix. On starts from a formula derived in [39] for the partition function:

$$Z = Z_0 + \frac{1}{4\pi i} \int dE e^{-\beta E} \text{Tr} \text{Im} \partial_E \log \widehat{S}(E) \quad (12)$$

where  $Z_0$  is the partition function for the free theory and  $\hat{S}(E)$  is the off-shell S-matrix operator in the usual scattering theory. Though the above expression is simple enough, a considerable amount of additional work is required to turn it into something useful. The trace is over the multi-particle Fock space, thus the above expression contains contributions from N-body processes for all N. For integrable theories in one spatial dimension, the N-body S-matrix factorizes into 2-body S-matrices, and though it has never been proven, the above expression should recover the thermodynamic Bethe ansatz[40].

As described in detail in [38], one can develop a diagrammatic description of the contributions to eq. (12), where vertices with  $2N$  legs correspond to the logarithm of the S-matrix for  $N$  particle scattering and lines connection the vertices are occupation numbers. The free energy can be obtained from a variational principle based on a Legendre transformation between the chemical potential and the density. The variational principle leads to an integral equation satisfied by the occupation numbers with kernels involving the logarithm of the S-matrix. In the present context, we restrict ourselves to 2-body processes only, which, as explained in the last section, can be calculated exactly. This should be a good approximation if the gas is not too dense. We also remark that this method is quite different than the method used for example by Nozieres-Schmitt-Rink[11], although both involve the S, or t-matrix. The latter formalism does not lead to the self-consistent integral equations we describe below, but rather a BCS-like equation. We point out that this latter method was extended to the unitary gas in [31].

Let us now describe the main result derived in [38]. Consider again for simplicity a single component bosonic or fermionic gas. The filling fractions, or occupation numbers, are parameterized in terms of a pseudo-energy  $\varepsilon(\mathbf{k})$ :

$$f(\mathbf{k}) = \frac{1}{e^{\beta\varepsilon(\mathbf{k})} - s} \tag{13}$$

which determine the density:

$$n = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{e^{\beta\varepsilon(\mathbf{k})} - s} \quad (14)$$

where  $s = 1, -1$  corresponds to bosons, fermions respectively. The consistent summation of 2-body scattering leads to an integral equation for the pseudo-energy  $\varepsilon(\mathbf{k})$ , analogous to the Yang-Yang integral equation. It is convenient to define the quantity:

$$y(\mathbf{k}) = e^{-\beta(\varepsilon(\mathbf{k}) - \omega_{\mathbf{k}} + \mu)} \quad (15)$$

where  $\omega_{\mathbf{k}} = \mathbf{k}^2/2m$ . Then  $y$  satisfies the integral equation

$$y(\mathbf{k}) = 1 + \beta \int \frac{d^3\mathbf{k}'}{(2\pi)^3} G(\mathbf{k} - \mathbf{k}') \frac{y(\mathbf{k}')^{-1}}{e^{\beta\varepsilon(\mathbf{k}')} - s} \quad (16)$$

The free energy density is then

$$\mathcal{F} = -T \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left[ -s \log(1 - se^{-\beta\varepsilon}) - \frac{1}{2} \frac{(1 - y^{-1})}{e^{\beta\varepsilon} - s} \right] \quad (17)$$

The kernel  $G$  is related to the logarithm of the 2-body S-matrix of the last section, and depends on the coupling  $g$ . In the unitary limit  $g \rightarrow g_*$  the kernel simplifies greatly since  $S = -1$ :

$$G(\mathbf{k} - \mathbf{k}') = \mp \frac{8\pi^2}{m|\mathbf{k} - \mathbf{k}'|}, \quad (18)$$

where the  $-$  sign corresponds to  $g$  being just below the fixed point  $g_*$ , where the scattering length  $a \rightarrow +\infty$  on the BEC side, whereas the  $+$  sign corresponds to  $a \rightarrow -\infty$  on the BCS side. As explained in the last section, we will work on the BCS side.

Finally comparing with the definitions above for the scaling functions  $c, q$  one finds:

$$q(x) = \frac{2}{\sqrt{\pi}} \int_0^\infty d\kappa \sqrt{\kappa} \frac{y(\kappa)z}{e^\kappa - sy(\kappa)z} \quad (19)$$

and

$$c = \frac{2}{\sqrt{\pi}\zeta(5/2)} \int_0^\infty d\kappa \sqrt{\kappa} \left( -s \log(1 - s z y(\kappa) e^{-\kappa}) - \frac{1}{2} \frac{z(y(\kappa) - 1)}{e^\kappa - s z y(\kappa)} \right) \quad (20)$$

where  $z = e^{\mu/T}$  is the fugacity and the dimensionless integration variable is  $\kappa = \mathbf{k}^2/2mT$ . The ideal, free gas limit corresponds to  $y = 1$  where  $q = s \text{Li}_{3/2}(sz)$  and  $c = s \text{Li}_{5/2}(sz)/\zeta(5/2)$ , where  $\text{Li}$  is the polylogarithm. The BEC critical point of the ideal gas occurs at  $\mu = 0$ , i.e.  $q = \zeta(3/2)$ .

Consider now two-component fermions with the action (3). There are two pseudo-energies  $\varepsilon_{\uparrow,\downarrow}$  satisfying two coupled integral equations. Due to the  $\text{SU}(2)$  symmetry, for equal chemical potentials  $\varepsilon_\uparrow = \varepsilon_\downarrow$ . However the available phase space for 2-particle scattering is doubled and the kernels have an extra  $1/2$ :

$$G_{\text{fermi}} = \frac{1}{2} G_{\text{bose}} \quad (21)$$

Thus the two-component fermion reduces to two identical copies of the above 1-component expressions, with the modification (21).

#### IV. PHASE TRANSITIONS IN TEMPERATURE

In the present context of quantum statistical mechanics, the quantum critical point at  $g = g_*$  does not signify a phase transition. However in the unitary limit at fixed  $g = g_*$  phase transitions may occur in regions of temperature versus density. From the scaling form of the free energy, such a phase transition must occur at a fixed, specific value of  $x = x_c = \mu/T$ . We will continue to refer to such phase transitions as critical points. The simplest case is BEC in the free boson theory, where the phase transition occurs at  $x_c = 0$ , which can be translated into the well-known relation between temperature and density at the critical point  $n_c \Lambda_T^3 = \zeta(3/2) = 2.61$ .

### A. Fermionic case

The integral equation for  $y(\kappa)$ , eq. (16), can be solved numerically by iteration. One first substitutes  $y_0 = 1$  on the right hand side and this gives the approximation  $y_1$  for  $y$ . One then substitutes  $y_1$  on the right hand side to generate  $y_2$ , etc. For regions of  $z$  where there are no critical points, this procedure converges rapidly, and as little as 5 iterations are needed. For fermions, as one approaches zero temperature, i.e.  $x$  large and positive, more iterations are needed for convergence. The following results are based on 50 iterations.

When  $z \ll 1$ ,  $y \approx 1$ , and the properties of the free ideal gas are recovered, since the gas is very dilute. There are solutions to eq. (16) for all  $-\infty < x < \infty$ . ( $x = \mu/T$ ). The scaling function  $c$  at zero chemical potential is  $c(0) = 1.76$  compared to the free fermion value  $c_{\text{free}} = 2 - 2^{-1/2} = 1.29$ , thus the interactions have a significant effect. (These are twice the 1-component values.)

Whereas  $c$  and  $q$  are nearly featureless, other quantities seem to indicate a phase transition at large density. For instance, the entropy per particle decreases with decreasing temperature up to  $x < x_c \approx 11.2$ . Beyond this point the entropy per particle has the unphysical behavior of increasing with temperature. A further indication that the region  $x > x_c$  is unphysical is that the specific heat per particle becomes negative, as shown in Figure 1. When  $x \ll 0$ ,  $C_V/N$  approaches the classical value  $3/2$ . This leads us to suggest a phase transition, at  $x = x_c = 11.2$ . This value of  $x$  can be expressed as a critical temperature  $T_c$  in units of the Fermi energy  $\epsilon_F = k_B T_F$ . From the definition  $\epsilon_F = (3\pi^2 n / \sqrt{2})^{2/3} / m$ , one has  $T/T_F = (4/3\sqrt{\pi}q)^{2/3}$ . From the value  $q(x_c)$ , one finds the critical temperature  $T_c/T_F \approx 0.1$ . As we will show, our analysis of the viscosity to entropy-density ratio suggests a higher  $T_c/T_F$ . There have been numerous estimates of  $T_c/T_F$  based on various approximation schemes, mainly using Monte Carlo methods on the lattice [17–22], quoting results for  $T_c/T_F$  between 0.05

and 0.23. The work [18] puts an upper bound  $T_c/T_F < 0.14$ , and the most recent results of Burovski et. al. quote  $T_c/T_F = 0.152(7)$ . Our result is thus consistent with previous work.

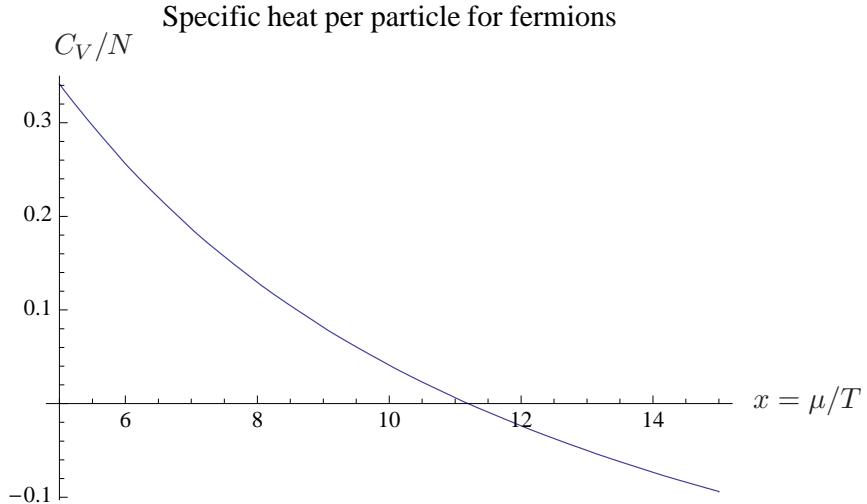


FIG. 1: Specific heat per particle as a function of  $x$  for fermions.

### B. Bosonic case

The possibility of a phase transition in the unitary Bose gas is more subtle, since a bosonic gas with attractive interactions is susceptible to mechanical collapse to a denser state. This issue has been studied for small negative scattering length in a number of works[34–37], and the consensus is that mechanical collapse occurs before BEC. However it is not clear whether this conclusion can be extrapolated to unitary limit where the scattering length is infinite. In the approach described in the last section, we found strong evidence for a phase transition to a strongly interacting version of BEC, as we now explain.

We again solved the integral equation (16) by iteration, starting from  $y = 1$ . Since the occupation numbers decay quickly as a function of  $\kappa$ , we introduced a cut-off  $\kappa < 10$ . For  $x$  less than approximately  $-2$ , the gas behaves nearly classically. The main feature of the solution to the integral equation is that for  $x > x_c \equiv -1.2741$ , there is no solution that is smoothly connected to the classical limit  $x \rightarrow -\infty$ . Numerically, when there is no solution the iterative procedure fails to converge. In Figure 2, we plot  $\varepsilon(\mathbf{k} = 0)$  as a function of  $x$ , and one sees that it goes to zero at  $x_c$ . This implies the occupation number  $f$  diverges at  $\mathbf{k} = 0$  at this critical point. One clearly sees this behavior in Figure 3. We also found that the compressibility diverges at  $x_c$ , again consistent with BEC.

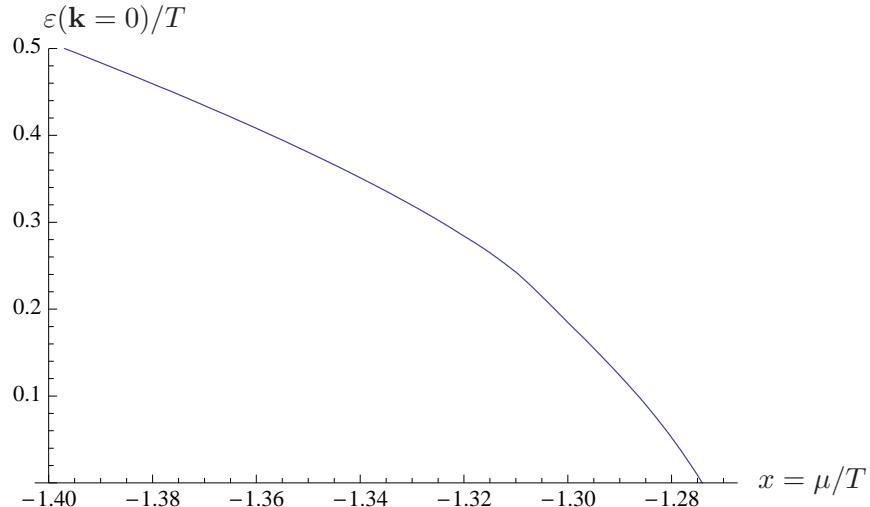


FIG. 2: The pseudo-energy  $\varepsilon$  at  $\mathbf{k} = 0$  as a function of  $x = \mu/T$ .

This strongly suggests that there is a critical point at  $x_c$  which is a strongly interacting, scale invariant version of the ideal BEC. In terms of the density, the critical point is:

$$n_c \lambda_T^3 = 1.325, \quad (\mu/T = x_c = -1.2741) \quad (22)$$

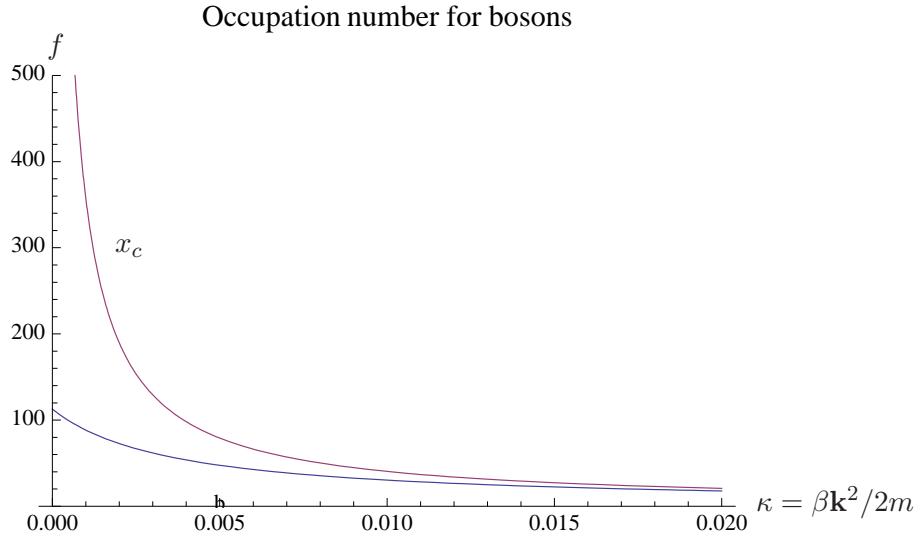


FIG. 3: The occupation number  $f(\kappa)$  for  $x = -1.275$  and  $x_c = -1.2741$ .

The negative value of the chemical potential is consistent with the effectively attractive interactions. The above should be compared with the ideal BEC of the free theory, where  $x_c = 0$  and  $n_c \lambda_T^3 = \zeta(3/2) = 2.61$ , which is higher by a factor of 2.

A critical exponent  $\nu$  characterizing the diverging compressibility can be defined as

$$\kappa \sim (T - T_c)^{-\nu} \quad (23)$$

A log-log plot of the compressibility versus  $T - T_c$  shows an approximately straight line, and we obtain  $\nu \approx 0.69$ . This should be compared with BEC in an ideal gas, where  $\nu \approx 1.0$ . Clearly the unitary gas version of BEC is in a different universality class.

## V. VISCOSITY TO ENTROPY DENSITY RATIO

Finally we turn to the intended focus of this article, the ratio of the viscosity to entropy density. Consider first a single component gas. The simplest expressions for the shear viscosity are based on kinetic theory, where it is related to the momentum transfer through an imaginary 2-dimensional plane cutting through the 3-dimensional bulk in the presence of a velocity gradient in the fluid flow. It can be expressed as

$$\eta = \frac{1}{3} n \bar{v} m \ell_{\text{free}} \quad (24)$$

where  $\bar{v}$  is the average speed and  $\ell_{\text{free}}$  is the mean free path[41]. The mean free path is  $\ell_{\text{free}} = 1/(\sqrt{2}n\sigma)$  where  $\sigma$  is the total cross-section. (The  $\sqrt{2}$  comes from the ratio of the mean speed to the mean relative speed.)

Consider first the bosonic model. (There were two errors by factors of 2 in the original discussion presented in [29] which are here corrected. However they turn out to cancel and the main result below eq. (28) is the same as before. ) In the unitary limit where the S-matrix  $S = -1$ , the amplitude is [29]:

$$\mathcal{M} = \frac{16\pi i}{m|\mathbf{k} - \mathbf{k}'|} \quad (25)$$

The cross section was calculated by the standard manner in which it is related to an integration over final state momenta and the square of the amplitude  $\mathcal{M}$

$$\sigma = \frac{m^2 |\mathcal{M}|^2}{8\pi} = \frac{32\pi}{|\mathbf{k} - \mathbf{k}'|^2} \quad (26)$$

Finally,

$$(|\mathbf{k} - \mathbf{k}'|^2)_{\text{ave}} = (\mathbf{k}^2 + \mathbf{k}'^2 - 2\mathbf{k} \cdot \mathbf{k}')_{\text{ave}} = 2m^2 \bar{v}^2 \quad (27)$$

since the cosine of the angle between  $\mathbf{k}$  and  $\mathbf{k}'$  averages to zero. This gives

$$\eta_{\text{boson}} = \frac{m^3 \bar{v}^3}{48\sqrt{2}\pi} \quad (28)$$

For a unitary gas, the relation between the energy, volume and pressure is the same as for a free theory:  $E/V = 3p/2$ [13]. Since the pressure is due to the kinetic energy, this implies

$$\frac{1}{2}m\bar{v}^2 = E/N = \frac{3}{2}\frac{c}{c'}T \quad (29)$$

Thus  $\eta \propto (cT/c')^{3/2}$ . In the high temperature limit, the function  $y \rightarrow 1$  and  $c \propto Li_{5/2}(e^x) \propto x$  and  $c' \propto Li_{3/2}(e^x) \propto x$  as  $x \rightarrow 0$ . Thus, as expected  $\eta \propto T^{3/2}$ . Since the entropy density  $s = -\partial\mathcal{F}/\partial T$ , one finally has

$$\frac{\eta}{s} = \frac{\sqrt{3\pi}}{8\zeta(5/2)} \left(\frac{c}{c'}\right)^{3/2} \frac{1}{5c/2 - xc'} \quad (30)$$

For two-component fermions, the available phase space, referred to as  $\mathcal{I}$  in [29], is doubled. Also, spin up particles only scatter with spin down. This implies the cross section is 8 times smaller than the expression (26). Since the entropy density is doubled due to the two components, this implies that  $\eta/s$  is 4 times the expression eq. (30), where  $c$  is the 1-component value appropriate to fermions.

The ratio  $\eta/s$  for fermions as a function of  $T/T_F$  is shown in Figure 4, and is in good agreement both quantitatively and qualitatively with the experimental data summarized in [6]. The lowest value occurs at  $x = 2.33$ , which corresponds to  $T/T_F = 0.28$ , and

$$\frac{\eta}{s} > 4.72 \frac{\hbar}{4\pi k_B} \quad (31)$$

This is consistent with the most recent experimental data which shows a minimum that is about 4 – 5 times the bound[32]. Other, considerably more complicated theoretical approaches, e.g. using a Kubo formula for the viscosity, gives results around 6 – 7 times the bound[42].

For bosons, the ration  $\eta/s$  is plotted in Figure 5 as a function of  $T/T_c$ . One sees that it has a minimum at the critical point, where

$$\frac{\eta}{s} > 1.26 \frac{\hbar}{4\pi k_B} \quad (32)$$

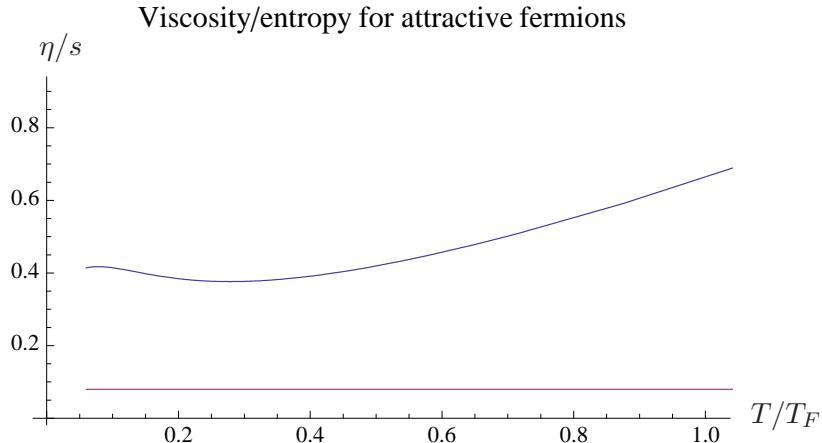


FIG. 4: The viscosity to entropy-density ratio as a function of  $T/T_F$  for fermions. The horizontal line is  $1/4\pi$ .

Thus the bosonic gas at the unitary critical point is a more perfect fluid than that of fermions.

## VI. CONCLUSIONS

In summary, we have calculated the ratio of viscosity to entropy density for both Bose and Fermi gases in the unitary limit using the S-matrix based approach in [29]. For the fermionic gas, we found  $\eta/s \geq$  is greater than 4.7 times the conjectured lower bound of  $1/4\pi$ , which is lower than the result of other theoretical approaches, and more consistent with the most recent experimental results[32]. We provided evidence that the unitary Bose gas is stable and has a strongly interacting BEC phase transition. The same calculation indicates that the Bose case is a more perfect fluid, with  $\eta/s \geq 1.7$  times the lower bound.

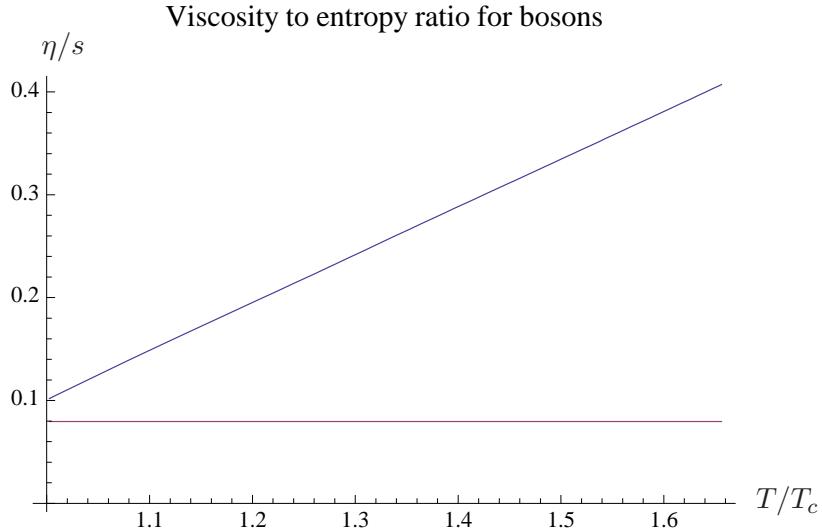


FIG. 5: The viscosity to entropy-density ratio as a function of  $T/T_c$  for bosons. The horizontal line is  $1/4\pi$ .

## VII. ACKNOWLEDGMENTS

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